

ON THE 3-DIMENSIONAL INVARIANT FOR CYCLIC CONTACT BRANCHED COVERINGS

TETSUYA ITO

ABSTRACT. We give a formula of 3-dimensional invariant for a cyclic contact branched covering of the standard contact S^3 .

1. INTRODUCTION

Let $\widetilde{M} \rightarrow M$ be a branched covering of a 3-manifold M , branched along a link $K \subset M$. When M has a contact structure ξ and K is a transverse link in the contact 3-manifold (M, ξ) , \widetilde{M} has the natural contact structure $\widetilde{\xi}$. We call the contact 3-manifold $(\widetilde{M}, \widetilde{\xi})$ the *contact branched covering* of (M, ξ) , branched along the transverse link K .

Let (M, ξ) be a p -fold cyclic contact branched covering of (S^3, ξ_{std}) (the standard contact S^3), branched along a transverse link K . In [5, Theorem 1.4], it is shown that the euler class $e(\xi)$ is zero, and the 3-dimensional invariant $d_3(\xi) \in \mathbb{Q}$ (See [3] for definition) only depends on a topological link type of K and its self-linking number.

In this note, we show a direct formula of $d_3(\xi)$ in terms of its branch locus K .

Theorem 1.1. *If a contact 3-manifold (M, ξ) is a p -fold cyclic contact branched covering of (S^3, ξ_{std}) , branched along a transverse link K , then*

$$d_3(\xi) = -\frac{3}{4} \sum_{\omega: \omega^p=1} \sigma_\omega(K) - \frac{p-1}{2} sl(K) - \frac{1}{2}p.$$

Here $\sigma_\omega(K)$ denotes the Tristram-Levine signature, the signature of $(1-\omega)A + (1-\bar{\omega})A^T$, where A denotes the Seifert matrix for K , and $sl(K)$ denotes the self-linking number.

Thus, our formula tells us that $d_3(\xi)$ actually only depends on the concordance class of K and the self-linking number. Also, by slice Bennequin inequality [7], it also shows that the smooth 4-genus $g_4(K)$ of K gives a lower bound of $d_3(\xi)$.

Corollary 1.2. *If a contact 3-manifold (M, ξ) is a p -fold cyclic contact branched covering of (S^3, ξ_{std}) branched along K , then $d_3(\xi) \geq -\frac{5}{2}(p-1)g_4(K) - \frac{1}{2}$.*

2. PROOF

Proof of Theorem 1.1. Let (M, ξ) be a p -fold cyclic contact branched covering, branched along a transverse link K in (S, ξ_{std}) . We put the transverse link K as a closed braid, the closure of an m -braid α .

Let (S, ψ) be the open book decomposition of (S^3, ξ_{std}) , whose binding is the (p, m) -torus link. Inside S^3 , the page S is an obvious Seifert surface of the (p, m) -torus link which we view as the closure of the p -braid $(\sigma_1 \cdots \sigma_{m-1})^p$ as we illustrate in Figure 1.

Topologically, the page S is the p -fold cyclic branched covering of the disk D^2 , branched along m -points. Let $\pi: B_m = MCG(D^2 \setminus \{m \text{ points}\}) \rightarrow MCG(S)$ be the map induced by the branched covering map, which is explicitly is written by $\pi(\sigma_i) = D_{i,1} \cdots D_{i,p-1}$ [5, Lemma 3.1]. Here $D_{i,j}$ denotes the right-handed Dehn twist along the curve $C_{i,j}$ on S , given in Figure 1. (Here we are assuming that $MCG(S)$ acts on S from left, so $D_{i,1} \cdots D_{i,p-1}$ means $D_{i,p-1}$ comes first and $D_{i,1}$ last.)

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An important observation is that in (S^3, ξ_{std}) , the curves $C_{i,j}$ are realized as the Legendrian unknot with $tb = -1, rot = 0$.

By using $D_{i,j}$, ψ is written by

$$\psi = \pi(\sigma_{m-1} \cdots \sigma_2 \sigma_1) = (D_{m-1,1} \cdots D_{m-1,p-1}) \cdots (D_{2,1} \cdots D_{2,p-1})(D_{1,1} \cdots D_{1,p-1}).$$

Also, $(S, \phi = \pi(\alpha))$ gives the open book decomposition of (M, ξ) .

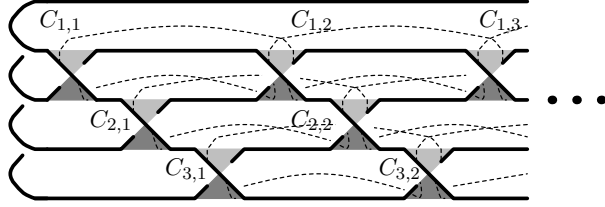


FIGURE 1. Page S of the open book (S, ψ) inside S^3 .

First we draw the surgery diagram of (M, ξ) from its open book decomposition (S, ϕ) , following the discussion in [5, Section 3]. We take a factorization of the braid $(\sigma_1^{-1} \cdots \sigma_{m-1}^{-1})\alpha$

$$(2.1) \quad (\sigma_1^{-1} \cdots \sigma_{m-1}^{-1})\alpha = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_n}^{\varepsilon_n} \quad (\varepsilon_j \in \{\pm 1\}, i_j \in \{1, \dots, m-1\})$$

by the standard generators $\{\sigma_1^{\pm 1}, \dots, \sigma_{m-1}^{\pm 1}\}$ of B_m . By replacing each $\sigma_i^{\pm 1}$ in (2.1) with the sequence of Dehn twists $(D_{i,1} \cdots D_{i,p-1})^{\pm 1}$, we have the factorization of $\psi^{-1}\phi$ by Dehn twists $D_{i,j}^{\pm 1}$,

$$(2.2) \quad \psi^{-1}\phi = \prod_{j=1}^n (D_{i_j,1} \cdots D_{i_j,p-1})^{\varepsilon_j}.$$

For each Dehn twist $D_{i,j}^{\pm 1}$ in the factorization (2.2) we put a curve $C_{i,j}$ on distinct pages on the open book (S, ψ) , so that it is a Legendrian unknot with $tb = -1, rot = 0$ in (S^3, ξ_{std}) . Then (M, ξ) is obtained by the contact surgery along the resulting Legendrian link. Here the surgery coefficient of a component is (-1) (resp. $(+1)$) if it comes from a positive (resp. negative) Dehn twist.

The factor σ_i in the factorization (2.1) gives a sequence of Dehn twists $(D_{i,1} \cdots D_{i,p-1})$ in the factorization (2.2). The Legendrian curves $C_{i,1}, \dots, C_{i,p-1}$, put in different pages (so that $C_{i,p-1}$ comes first and $C_{i,1}$ last), produce the $(p-1)$ component Legendrian link as we draw in Figure 2 (a). Similarly, σ_i^{-1} in the factorization (2.1) gives a sequence of Dehn twists $D_{i,p-1}^{-1} \cdots D_{i,1}^{-1}$ in the factorization (2.2), which produce the $(p-1)$ component Legendrian unlink as we draw in Figure 2 (b).

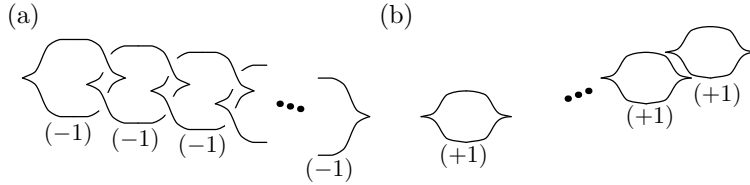


FIGURE 2. The contribution of $\sigma_i^{\pm 1}$ in the resulting contact surgery diagram

These local contributions of surgery links interact each other, whose linking patterns can be chased by looking the page S in Figure 1, as we summarize as follows (cf. [5, Fig 11, Remark 3.3]):

Observation 1. Let $\mathcal{L}_{i_k}^{\varepsilon_k} = C_{i_k,1} \cup \dots \cup C_{i_k,p-1}$ and $\mathcal{L}_{i_l}^{\varepsilon_l} = C_{i_l,1} \cup \dots \cup C_{i_l,p-1}$ be the sub Legendrian links in the contact surgery diagram of M , that comes from the k -th factor $\sigma_{i_k}^{\varepsilon_k}$ and l -th factor $\sigma_{i_l}^{\varepsilon_l}$ in the factorization (2.1), with $k < l$.

Then the components $C_{i_k,s}$ and $C_{i_l,t}$ link forms a (topological) positive Hopf link, if and only if $i_k \in \{i_l, i_l + 1\}$. Otherwise, two components $C_{i_k,s}$ and $C_{i_l,t}$ are disjoint. (See Figure 3.)

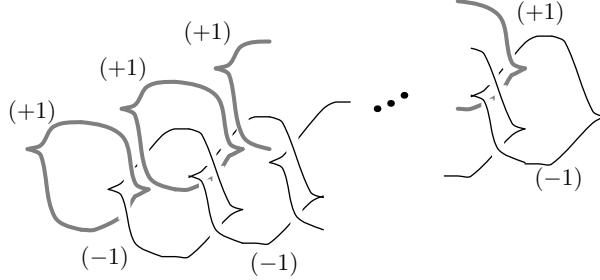


FIGURE 3. How the local contribution of $\sigma_i^{\pm 1}$ in the contact surgery diagram links each other. Here we illustrate contributions for σ_{i_k} (depicted by black line) and $\sigma_{i_l}^{-1}$ (depicted by gray line) with $k < l$ in the factorization (2.1), for the case $i_k \in \{i_l, i_l + 1\}$.

The contact surgery diagram provides a 4-manifold X that bounds M . By [2, Corollary 3.6],

$$d_3(\xi) = \frac{1}{4}(-3\sigma(X) - 2\chi(X)) + q,$$

where q is the number of $(+1)$ -contact surgeries, and $\chi(X)$ is the euler characteristic of X . Note that the term c^2 in the formula [2, Corollary 3.6] disappears since each component of the surgery link has zero rotation number. Let e_+ and e_- be the number of positive and negative Dehn twist in the factorization (2.1). Since each factor $\sigma_i^{\pm 1}$ produces $(p-1)$ (∓ 1) contact surgeries along unknots,

$$\begin{aligned} d_3(\xi) &= -\frac{3}{4}\sigma(X) - \frac{1}{2}((p-1)e_+ + (p-1)e_- + 1) + (p-1)e_- \\ &= -\frac{3}{4}\sigma(X) - \frac{p-1}{2}(e_+ - e_-) - \frac{1}{2}. \end{aligned}$$

By Bennequin's formula $sl(K) = e_+ - e_- + 1$, hence

$$(2.3) \quad d_3(\xi) = -\frac{3}{4}\sigma(X) - \frac{p-1}{2}sl(K) - \frac{p}{2}.$$

It remains to compute $\sigma(X)$. Take a factorization of the braid α given by

$$(2.4) \quad \alpha = \sigma_{m-1} \cdots \sigma_1 \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_n}^{\varepsilon_n} \quad (\varepsilon_j \in \{\pm 1\}, i_j \in \{1, \dots, m-1\}).$$

Let $\Sigma \subset S^3 = \partial B^4$ be the canonical Seifert surface of K that comes from the factorization (2.4). Namely, Σ is made of m disks $\{D_1, \dots, D_m\}$, with twisted bands connecting i -th and $(i+1)$ -st disk for each $\sigma_i^{\pm 1}$ in the factorization (2.1).

The following is the most crucial observation in our computation.

Claim 2.1. *Let W be the p -fold cyclic branched covering of B^4 branched along Σ (pushed into the interiors of B^4). Then X is diffeomorphic to W .*

Proof of Claim. We draw a Kirby diagram of W , following [1, Section 2] (see also [4, Section 6.3]).

Take a handle decomposition of Σ so that the 0-handle is $D_1 \cup h_1 \cup D_2 \cup \dots \cup h_{m-1} \cup D_m$, where h_i is the twisted band coming from the $(m-i)$ -th factor σ_i of the factorization (2.4), and that the 1-handles are the rest of twisted bands. We put Σ in the 3-space so that the 0-handle is the unit disc in the x - y plane, and that 1-handles are contained in the upper half-space. Then the

Kirby diagram of W is obtained by “symmetrizing” the cores of 1-handles of Σ whose framings are determined by the framings of the core of 1-handles. Except the simplest case $p = 2$, which we will explicitly illustrate later in Example 2.2, the diagram is complicated and it is not easy to draw the whole diagram – however, the contribution of 1-handle in the resulting Kirby diagram, and how they interact each other is simple. See Figure 4 below.

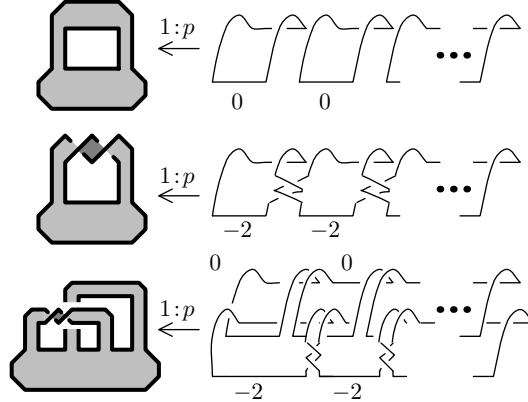


FIGURE 4. Branched covering of Seifert surface: 1-handle contribution, and how these contributions interact each other (when they are nested).

To put Σ in such a convenient position, first we flip the 1st disc D_1 , by untwisting the band h_1 (see Figure 5(a) –(d)). This simplifies the 0-handle of Σ , and iterating this flipping procedure, eventually we put Σ in such a convenient position (see Figure 6). In this process, all 1-handles gains negative half twist, so in the final position, the framing of 1-handle is (-1) if it comes from positive generator, and is 0 if it comes from negative generator.

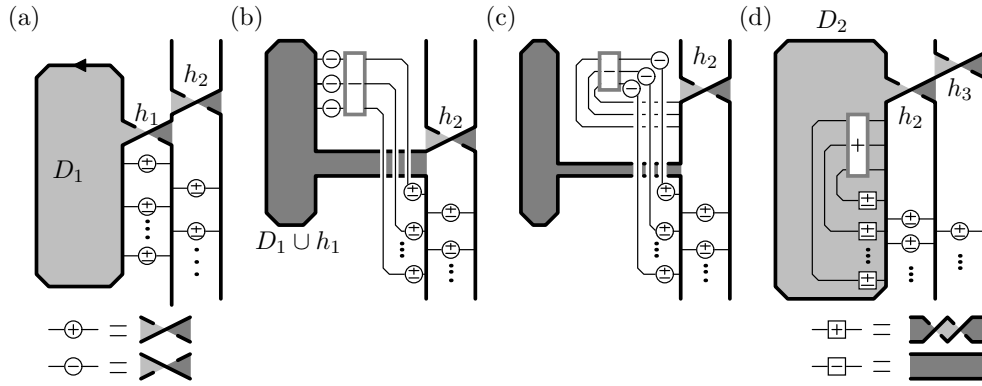


FIGURE 5. Putting the Seifert surface Σ into a nice position, by flipping D_1 along h_1 . Here we draw 1-handle by a line with \pm -sign coming from corresponding generator σ_i^\pm . The gray box inside \pm represents positive and negative half twist.

From this procedure, we observe:

Observation 2. The 1-handles h_k and h_l of Σ , coming from the k -th and l -th factor $\sigma_{i_k}^{\varepsilon_k}$ and $\sigma_{i_l}^{\varepsilon_l}$ in the factorization (2.1) ($k < l$), nest each other in Figure 6 if and only if $i_k \in \{i_l, i_l + 1\}$ (see Figure 7)

Recall that each component of the contact surgery diagram of M has $tb = -1$, so (-1) and $(+1)$ contact surgery corresponds to -2 and 0 topological surgery, respectively. Hence each factor

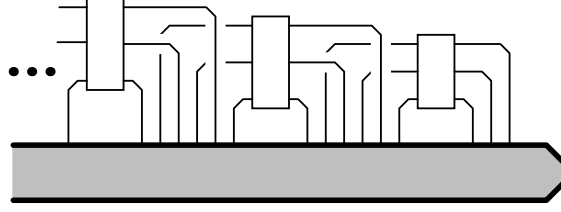


FIGURE 6. The canonical Seifert surface Σ , put in a convenient position for drawing Kirby diagram. A box represents the positive half twist, and each 1-handle depicted by line has either 0 or -1 framing.

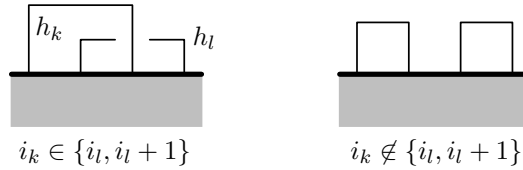


FIGURE 7. How 1-handles of Σ nest each other, when we put Σ in a convenient position as in Figure 6.

$\sigma_i^{\pm 1}$ in (2.1) contributes the same $(p - 1)$ components framed link in the surgery diagram of X and W (compare Figure 2 and Figure 4). Moreover, from Observation 1 and Observation 2, these local contributions link other part of the diagrams, in exactly the same way (compare Figure 3 and Figure 4). Thus, comparisons of the construction of the surgery diagrams for X and W proves that they are completely the same diagram. \square

Claim 2.1, together with a well-known fact on Tristram-Levine signature (see [6, Theorem 12.6], for example) shows

$$(2.5) \quad \sigma(X) = \sigma(W) = \sum_{\omega: \omega^p=1} \sigma_{\omega}(K).$$

The equalities (2.3) and (2.5) completes the proof. \square

Example 2.2 (The case double branched covering). In the case $p = 2$, the contact double branched covering, it is much easier to treat and draw the surgery diagram of X and W . Here we give more explicit illustrations of surgery diagrams.

Let (M, ξ) be a contact double branched covering branched along the closure of an m -braid α . We begin with the open book (S, ψ) whose binding is $(m, 2)$ -torus link. To visualize its symmetry, we view the the page S as the $(m - 1)$ -times plumbing of an annulus A_i that is the boundary of the positive Hopf link, as illustrated in Figure 8. As an element of the mapping class group of S , the standard generator σ_i lifts to the right-handed Dehn twist along the core of an annulus A_i .

By taking a factorization of the braid α , following the discussion in the proof of Theorem 1.1, we get a contact surgery diagram of (M, ξ) , as we draw in Figure 9. On the other hand, the Kirby diagram of W is obtained by “doubling” the core of 1-handles of the canonical Seifert surface Σ , as we show in Figure 10.

Now one immediately see that these two diagrams are the same.

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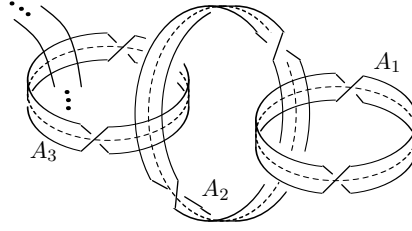


FIGURE 8. Page S of an open book of (S^3, ξ_{std}) whose binding is the $(m, 2)$ -torus link.

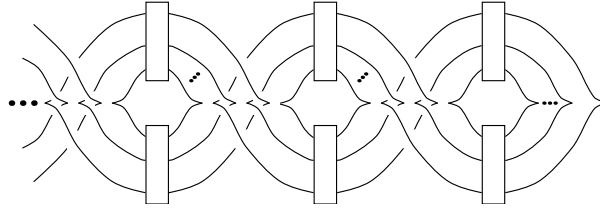


FIGURE 9. A contact surgery diagram of contact double branched covering. A box represents the positive half twist.

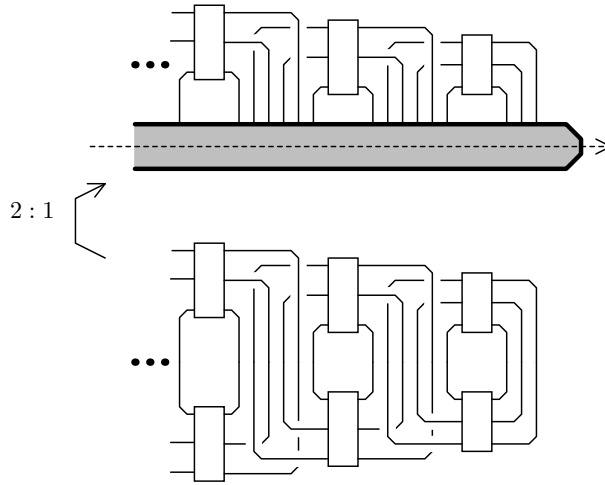


FIGURE 10. Kirby Diagram for double branched covering along the canonical Seifert surface Σ .

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO, 606-8502, JAPAN

E-mail address: tetito@kurims.kyoto-u.ac.jp

URL: <http://www.kurims.kyoto-u.ac.jp/~tetito/>